ALGEBRA I

M. Math. I

Final Examination

Instructions: All questions carry equal marks.

1. (a) State Gauss Lemma for Unique Factorisation Domains.

   (b) Prove that the polynomial X^4-10X^2+1 is irreducible over integers but can be factored modulo every prime p.

2. Let f, g be polynomials in $\mathbb{C}[X, Y]$ with no common factors. Prove that the ring $R = \mathbb{C}[X, Y]/(f, g)$ is a finite-dimensional vector space over $\mathbb{C}$. Is the converse also true? Justify your answer.

3. Explain in detail the class equation of a finite group G. Let G be a non-abelian group of order p^3 where p is a prime number.

   (a) What are the possible orders of its center $Z(G)$?

   (b) What are the possible class equations of G?

4. Let p, q be prime numbers such that $q|(p-1)$. Show that there exists a subgroup of order pq of the permutation group S_p. Can this group be abelian? Justify your answer.

5. Prove that no group of order 24 is simple.